Specific features of differential equations of mathematical physics

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Abstract

Three types of equations of mathematical physics, namely, the equations, which describe any physical processes, the equations of mechanics and physics of continuous media, and field-theory equations are studied in this paper.

In the first and second case the investigation is reduced to the analysis of the nonidentical relations of the skew-symmetric differential forms that are obtained from differential equations. It is shown that the integrability of equations and the properties of their solutions depend on the realization of the conditions of degenerate transformations under which the identical relations are obtained from the nonidentical relation.

The field-theory equations, in contrast to the equations of first two types, are the relations made up by skew-symmetric differential forms or their analogs (differential or integral ones). This is due to the fact that the field-theory equations have to describe physical structures (to which closed exterior forms correspond) rather than physical quantities. The equations that correspond to field theories are obtained from the equations that describe the conservation laws (of energy, linear momentum, angular momentum, and mass) of material systems (of continuous media). This disclose a connection between field theories and the equations for material systems (and points to that material media generate physical fields).

1. Specific features of equations descriptive of physical processes

Specific features of differential equations descriptive of physical processes and the types of their solutions can be demonstrated by the example of first-order partial differential equation using the properties of skew-symmetric differential forms.

[The method of investigating differential equations using skew-symmetric differential forms was developed by Cartan [1] in his analysis of the integrability of differential equations. Here we present this analysis to demonstrate specific features of differential equations and properties of solutions to these equations.]

Let

$$F(x^i,\, u,\, p_i) = 0, \quad p_i \,=\, \partial u/\partial x^i \tag{1}$$

be a first-order partial differential equation.

Let us consider the functional relation

$$du \,=\, \theta \tag{2}$$

where $\theta \,=\, p_i\, dx^i$ (the summation over repeated indices is implied). Here $\theta \,=\, p_i\, dx^i$ is a differential form of the first degree.

The specific feature of functional relation (2) is that in the general case, when differential equation (1) describes any physical processes, this relation turns out to be nonidentical one.

The left-hand side of this relation involves a differential, and the right-hand side includes the differential form $\theta = p_i\,dx^i$. For this relation be identical, the differential form $\theta = p_i\,dx^i$ must also be a differential (like the left-hand side of relation (2)), that is, it has to be a closed exterior differential form. To do this, it requires the commutator $K_{ij} = \partial p_j/\partial x^i - \partial p_i/\partial x^j$ of the differential form θ has to vanish.

In the general case from equation (1) it does not follow (explicitly) that the derivatives $p_i = \partial u/\partial x^i$, which obey to the equation (and given boundary or initial conditions of the problem), make up a differential. For equations descriptive of any processes (without any supplementary conditions), the commutator K_{ij} of the differential form θ is not equal to zero. The form $\theta = p_i\,dx^i$ turns out to be unclosed and is not a differential like the left-hand side of relation (2). Functional relation (2) appears to be nonidentical.

The nonidentity of functional relation (2) points to a fact that without additional conditions the derivatives of original equation do not make up a differential. This means that the corresponding solution u of the differential equation will not be a function of only variables x^i. The solution will depend on the commutator of the form θ, that is, it will be a functional.

To obtain a solution that is a function (i.e., the derivatives of this solution make up a differential), it is necessary to add the closure condition for the form $\theta = p_i dx^i$ and for relevant dual form (in the present case the functional F plays a role of a form dual to θ) [1]:

$$\left\{\begin{array}{l} dF(x^i,\,u,\,p_i)\;=\;0 \\ d(p_i\,dx^i)\;=\;0 \end{array}\right. \tag{3}$$

If we expand the differentials, we get a set of homogeneous equations with respect to dx^i and dp_i (in the $2n$-dimensional tangent space):

$$\left\{\begin{array}{l} \left(\dfrac{\partial F}{\partial x^i}\;+\;\dfrac{\partial F}{\partial u}\,p_i\right)\,dx^i\;+\;\dfrac{\partial F}{\partial p_i}\,dp_i\;=\;0 \\ dp_i\,dx^i\;-\;dx^i\,dp_i\;=\;0 \end{array}\right. \tag{4}$$

It is well-known that *vanishing the determinant* composed of coefficients at dx^i, dp_i is a solvability condition of the system of homogeneous differential equations. This leads to relations:

$$\frac{dx^i}{\partial F/\partial p_i}\;=\;\frac{-dp_i}{\partial F/\partial x^i + p_i \partial F/\partial u} \tag{5}$$

Relations (5) specify the integrating direction. namely, a pseudostructure, on which the form $\theta = p_i\,dx^i$ turns out to be closed one, i.e. it becomes a differential, and from relation (2) the identical relation is produced. One the pseudostructure, which is defined by relation(5), the derivatives of differential

equation (1) constitute a differential $\delta u = p_i dx^i = du$ (on the pseudostructure), and the means that the solution of equation (1) becomes a function.

Solutions, namely, functions on the pseudostructures formed by the integrating directions, are the so-called generalized solutions.

[If we find the characteristics of equation (1), it appears that relations (5) are characteristic relations [2]. That is, the characteristics are examples of the pseudostructures on which the derivatives of the differential equation made up closed forms and the solutions prove to be functions (generalized solutions).]

If the requirements of closure of skew-symmetric form made up by the derivatives of differential equation and relevant dual form are not fulfilled, that is, the derivatives do not form a differential, the solution corresponding to such derivatives will depend on the differential form commutator formatted by derivatives. That means that the solution is a functional rather than a function.

The first-order partial differential equation has been analyzed, and the functional relation with the form of the first degree has been considered.

Similar functional properties have the solutions to all differential equations describing physical processes. And, if the order of the differential equation is k, the functional relation with the k-degree form corresponds to this equation.

Thus one can see that the differential equations describing any physical fields can have solutions of two types, namely, generalized solutions which depend on variables only, and the solutions which are functionals since they depend on the commutator made up by mixed derivatives. A specific feature of generalized solutions consists in the fact that they can be realized only under *degenerate transformations*. The relations (5) corresponding to generalized solutions had been obtained the condition of *vanishing the determinant* composed of coefficients at dx^i, dp_i in the set of equations (4). This is a condition of *degenerate* transformation. (They are connected with symmetries of commutators of skew-symmetric forms.)

[It is clear that the degenerate transformation is a transition from tangent space to cotangent space (the Legendre transformations). The coordinates in relations (5) are not identical to the independent coordinates of the initial space on which equation (1) is defined.]

Since generalized solutions are possible only under realization of the conditions of degenerate transforms, they are discrete solutions (defined only on pseudostructures) and have discontinuities in the direction normal to pseudostructures.

The solutions being functionals disclose the another peculiarity of the solutions of differential equations, namely, their instability. The dependence of the solution on the commutator may lead to instability. The instability develops when the integrability conditions are not realized and exact (generalized) solutions are not formatted. (Thus, the solutions to the equations of the elliptic type may be unstable.)

[One can see that the qualitative theory of differential equations that solves the problem of unstable solutions and integrability bases on the properties nonidentical functional relation.]

2. Peculiarities of differential equations of mechanics and physics of material media

In analysis of partial differential equations the conjugacy of derivatives in different directions was studied (using the nonidentical functional relation). Under description of physical processes in material (continuous) media one obtains not one differential equation but a set of differential equations. And in this case it is necessary to investigate the conjugacy of not only derivatives in different directions but also the conjugacy (consistency) of the equations of this set. In this case from this set of equations one also obtains nonidentical relation that allows to study the conjugacy of equations and features of their solutions.

[The material (continuous) medium - material system - is a variety (infinite) of elements that have internal structure and interact among themselves. Thermodynamical, gasodynamical and cosmologic system, systems of elementary particles and others are examples of material system. (Physical vacuum can be considered as an analog of such material system.) Electrons, protons, neutrons, atoms, fluid particles and so on are examples of elements of material system.]

Equations of mechanics and physics of continuous media are equations that describe the conservation laws for energy, linear momentum, angular momentum and mass. Such conservation laws can be named as balance ones since they establish the balance between the variation of a physical quantity and corresponding external action.

The equations of balance conservation laws are differential (or integral) equations that describe a variation of functions corresponding to physical quantities [3-5]. (The Navier-Stokes equations are an example [5].)

(Mechanics and physics of continuous media treat the same equations. However an approach to solving these equations in mechanics and physics are different. Below it will be shown in what this difference manifest themselves.)

It appears that, even without a knowledge of the concrete form of these equations, one can see specific features of these equations and their solutions using skew-symmetric differential forms.

To do so it is necessary to study the conjugacy (consistency) of these equations.

The functions for equations of material media sought are usually functions which relate to such physical quantities like a particle velocity (of elements), temperature or energy, pressure and density. Since these functions relate to one material system, it has to exist a connection between them. This connection is described by the state-function. Below it will be shown that the analysis of integrability and consistency of equations of balance conservation laws for material media reduces to a study the nonidentical relation for the state-function.

Let us analyze the equations that describe the balance conservation laws for energy and linear momentum.

We introduce two frames of reference: the first is an inertial one (this frame of reference is not connected with the material system), and the second is an accompanying one (this system is connected with the manifold built by the trajectories of the material system elements).

The energy equation in the inertial frame of reference can be reduced to the form:

$$\frac{D\psi}{Dt} = A_1$$

where D/Dt is the total derivative with respect to time, ψ is the functional of the state that specifies the material system, A_1 is the quantity that depends on specific features of the system and on external energy actions onto the system. {The action functional, entropy, wave function can be regarded as examples of the functional ψ. Thus, the equation for energy presented in terms of the action functional S has a similar form: $DS/Dt = L$, where $\psi = S$, $A_1 = L$ is the Lagrange function. In mechanics of continuous media the equation for energy of an ideal gas can be presented in the form [5]: $Ds/Dt = 0$, where s is entropy.}

In the accompanying frame of reference the total derivative with respect to time is transformed into the derivative along the trajectory. Equation of energy is now written in the form

$$\frac{\partial\psi}{\partial\xi^1} = A_1 \tag{6}$$

Here ξ^1 is the coordinate along the trajectory.

In a similar manner, in the accompanying reference system the equation for linear momentum appears to be reduced to the equation of the form

$$\frac{\partial\psi}{\partial\xi^\nu} = A_\nu, \quad \nu = 2, ... \tag{7}$$

where ξ^ν are the coordinates in the direction normal to the trajectory, A_ν are the quantities that depend on the specific features of material system and on external force actions.

Eqs. (6) and (7) can be convoluted into the relation

$$d\psi = A_\mu \, d\xi^\mu, \quad (\mu = 1, \nu) \tag{8}$$

where $d\psi$ is the differential expression $d\psi = (\partial\psi/\partial\xi^\mu)d\xi^\mu$.

Relation (8) can be written as

$$d\psi = \omega \tag{9}$$

here $\omega = A_\mu \, d\xi^\mu$ is the skew-symmetrical differential form of the first degree.

Relation (9) has been obtained from the equation of the balance conservation laws for energy and linear momentum. In this relation the form ω is that of the first degree. If the equations of the balance conservation laws for angular momentum be added to the equations for energy and linear momentum, this form will be a form of the second degree. And in combination with the equation of the balance conservation law for mass this form will be a form of degree 3. In general case the evolutionary relation can be written as

$$d\psi = \omega^p \tag{10}$$

where the form degree p takes the values $p = 0, 1, 2, 3$. (The relation for $p = 0$ is an analog to that in the differential forms, and it was obtained from the interaction of energy and time.)

Since the balance conservation laws are evolutionary ones, the relations obtained are also evolutionary relations, and the skew-symmetric forms ω and ω^p are evolutionary ones.

Relations obtained from the equation of the balance conservation laws, as well as functional relation (2), turn out to be nonidentical.

To justify this we shall analyze relation (9). This relation proves to be nonidentical since the left-hand side of the relation is a differential, which is a closed skew-symmetric form, but the right-hand side of the relation involves the skew-symmetric differential form ω, which is unclosed form. The commutator made up by the derivatives of coefficients A_μ the form ω itself is also nonzero, since the coefficients A_μ are of different nature, that is, some coefficients have been obtained from the energy equation and depend on the energetic actions, whereas the others have been obtained from the equation for linear momentum and depend on the force actions.

In a similar manner one can prove the nonidentity of relation (10).

Nonidentity of the evolutionary relation, as well as nonidentity on functional relation (2), means that initial equations of balance conservation laws are not conjugated, and hence they are not integrable. The solutions of these equations can be functional or generalized ones. In this case generalized solutions are obtained only under degenerated transformations.

A type of solutions of the balance conservation law equations is essential to mechanics and physics of continuous media. And in physics and mechanics the interest is expressed in different types of solutions. In physics the interest is expressed in only generalized solutions that are invariant ones, and noninvariant solutions are ignored (even they have a physical meaning). In contrast to this, in mechanics of continuous media, where typically the equations are solved numerically, one searches for solutions that are functionals. In this case the question of searching for invariant solutions that are realized only under additional conditions is not posed.

Such limited approach to solving the equations of material media has some negative points. The physical approach enables one to find possible invariant solutions, however in this approach there is no way of telling in what instant of time of evolutionary process one or another solution is obtained. This does not also discloses the causality of phenomenon described by these solutions. The approach exploited in mechanics of continuous media leads to difficulties in explaining such phenomena as origination any discrete formations (like a generation of waves or turbulent pulsations, birth of massless particles and so on), to which the invariant solutions are assigned. (The answer to the questions arisen in physics and mechanics while solving the equations describing material media can be found by analysis of the nonidentical evolutionary relation obtained from these equations.)

The evolutionary relation obtained from equations of balance conservation laws for material systems (continuous media), in contrast to functional relation

(2), carries not only mathematical but also large physical loading [6,7]. This is due to the fact that the evolutionary relation possesses the duality. On the one hand, this relation corresponds to material system, and on other, as it will be shown below, describes the mechanism of generating physical structures. This discloses the properties and peculiarities of the field-theory equations and their connection with the equations of balance conservation laws.

Physical significance of nonidentical evolutionary relation.

The evolutionary relation describes the evolutionary process in material system since this relation includes the state differential, which specifies the material system state. However, since this relation turns out to be not identical, from this relation one cannot get the differential $d\psi$. The absence of differential means that the system state is nonequilibrium.

The evolutionary relation possesses one more peculiarity, namely, this relation is a selfvarying relation. (The evolutionary form entering into this relation is defined on the deforming manifold made up by trajectories of the material system elements. This means that the evolutionary form basis varies. In turn, this leads to variation of the evolutionary form, and the process of intervariation of the evolutionary form and the basis is repeated.)

Selfvariation of the nonidentical evolutionary relation points to the fact that the nonequilibrium state of material system turns out to be selfvarying. (It is evident that this selfvariation proceeds under the action of internal force whose quantity is described by the commutator of the unclosed evolutionary form ω^p.) State of material system changes but remains nonequilibrium during this process.

Since the evolutionary form is unclosed, the evolutionary relation cannot be identical. This means that the nonequilibrium state of material system holds. But in this case it is possible a transition of material system to a locally equilibrium state.

This follows from one more property of nonidentical evolutionary relation. Under selfvariation of the evolutionary relation it can be realized the conditions of degenerate transformation. And under degenerate transformation from the nonidentical relation it is obtained the identical relation.

From identical relation one can define the state differential pointing to the equilibrium state of the system. However, such system state is realized only locally due to the fact that the state differential obtained is an interior one defined only on pseudostructure, that is specified by the conditions of degenerate transformation. And yet the total state of material system remains to be nonequilibrium because the evolutionary relation, which describes the material system state, remains nonidentical one.

The conditions of degenerate transformation are connected with symmetries caused by degrees of freedom of material system. These are symmetries of the metric forms commutators of the manifold. {To the degenerate transformation it must correspond a vanishing of some functional expressions, such as Jacobians, determinants, the Poisson brackets, residues and others. Vanishing of these functional expressions is the closure condition for dual form. And it should be emphasize once more that the degenerate transformation is realized as a transition from the accompanying noninertial frame of reference

to the locally inertial system. The evolutionary form and nonidentical evolutionary relation are defined in the noninertial frame of reference (deforming manifold). But the closed exterior form obtained and the identical relation are obtained with respect to the locally-inertial frame of reference (pseudostructure)}.

Realization of the conditions of degenerate transformation is a vanishing of the commutator of manifold metric form, that is, a vanishing of the dual form commutator. And this leads to realization of pseudostructure and formatting the closed inexact form, whose closure conditions have the form

$$d_\pi \omega^p = 0, d_\pi{}^* \omega^p = 0 \tag{12}$$

On the pseudostructure π from evolutionary relation (10) it is obtained the relation

$$d_\pi \psi = \omega_\pi^p \tag{13}$$

which proves to be an identical relation since the closed inexact form is a differential (interior on pseudostructure).

The realization of the conditions of degenerate transformation and obtaining identical relation from nonidentical one has both mathematical and physical meaning. Firstly, this points to the fact that the solution of equations of balance conservation laws proves to be a generalized one. And secondly, from this relation one obtains the differential $d_\pi \psi$ and this points to the availability of the state-function (potential) and that the state of material system is in local equilibrium.

Relation (13) holds the duality. The left-hand side of relation (13) includes the differential, which specifies material system and whose availability points to the locally-equilibrium state of material system. And the right-hand side includes a closed inexact form, which is a characteristics of physical fields. The closure conditions (12) for exterior inexact form correspond to the conservation law, i.e. to a conservative on pseudostructure quantity, and describe a differential-geometrical structure. These are such structures (pseudostructures with conservative quantities) that are physical structures formatting physical fields[6].

The transition from nonidentical relation (10) obtained from the balance conservation laws to identical relation (13) means the following. Firstly, an emergency of the closed (on pseudostructure) inexact exterior form (right-hand side of relation (13)) points to an origination of the physical structure. And, secondly, an existence of the state differential (left-hand side of relation (13)) points to a transition of the material system from nonequilibrium state to the locally-equilibrium state.

Thus one can see that the transition of material system from nonequilibrium state to locally-equilibrium state is accompanied by originating differential-geometrical structures, which are physical structures. Massless particles, charges, structures made up by eikonal surfaces and wave fronts, and so on are examples of physical structures.

The duality of identical relation also explains the duality of nonidentical evolutionary relation. On the one hand, evoltionary relation describes the evolutionary process in material systems, and on the other describes the process of generating physical fields.

Such duality, which establishes the connection between material systems and physical fields, discloses one more peculiarity of evolutionary processes in material media.

The emergency of physical structures in the evolutionary process reveals in material system as an emergency of certain observable formations, which develop spontaneously. Such formations and their manifestations are fluctuations, turbulent pulsations, waves, vortices, and others. It appears that structures of physical fields and the formations of material systems observed are a manifestation of the same phenomena. The light is an example of such a duality. The light manifests itself in the form of a massless particle (photon) and of a wave.

This duality also explains a distinction in studying the same phenomena in material systems and physical fields. As it had already noted, in the physics of continuous media (material systems) the interest is expressed in generalized solutions of equations of the balance conservation laws. These are solutions that describe the formations in material media observed. The investigation of relevant physical structures is carried out using the field-theory equations.

The unique properties of nonidentical evolutionary relation, which describes the connection between physical fields and material systems, discloses the connection of evolutionary relation with the field-theory equations. In fact, all equations of existing field theories are the analog to such relation or its differential or tensor representation.

3. Specific features of field-theory equations

The field-theory equations are equations that describe physical fields. Since physical fields are formatted by physical structures, which are described by closed exterior *inexact* forms and by closed dual forms (metric forms of manifold), is obvious that the field-theory equations or solutions to these equations have to be connected with closed exterior forms. Nonidentical relations for functionals like wave-function, action functional, entropy, and others, which are obtained from the equations for material media (and from which identical relations with closed forms describing physical fields are obtained), just disclose the specific features of the field-theory equations.

The equations of mechanics, as well as the equations of continuous media physics, are partial differential equations for desired functions like a velocity of particles (elements), temperature, pressure and density, which correspond to physical quantities of material systems (continuous media). Such functions describe the character of varying physical quantities of material system. The functionals (and state-functions) like wave-function, action functional, entropy and others, which specify the state of material systems, and corresponding relations are used in mechanics and continuous media physics only for analysis

of integrability of these equations. And in field theories such relations play a role of equations. Here it reveals the duality of these relations. In mechanics and continuous media physics these equations describe the state of material systems, whereas in field-theory they describe physical structures from which physical fields are formatted.

{In differential equations of mathematical physics, which describe physical processes, the functions required are found by integrating derivatives obtained from the differential equation. And in field-theory equations the functions required follow not from derivatives, but from differentials of identical relations and they are exterior forms. That is, in mathematical physics one has to distinguish two types of differential equations, namely, the differential equations, which describe the variations of physical quantities, and the field-theory equations, which describe physical structures.}

It can be shown that all equations of existing field theories are in essence relations that connect skew-symmetric forms or their analogs (differential or tensor ones). And yet the nonidentical relations are treated as equations from which it can be found identical relation with include closed forms describing physical structures desired.

Field equations (the equations of the Hamilton formalism) reduce to identical relation with exterior form of first degree, namely, to the Poincare invariant

$$ds \,=-\,H\,dt\,+\,p_j\,dq_j \tag{14}$$

{The field equation has the form [2]

$$\frac{\partial s}{\partial t}+H\left(t,\,q_j,\,\frac{\partial s}{\partial q_j}\right)\,=\,0,\quad \frac{\partial s}{\partial q_j}\,=\,p_j \tag{15}$$

here s is a field function for the action functional $S\,=\,\int L\,dt$. Here L is the Lagrangian function, $H(t,\,q_j,\,p_j)\,=\,p_j\,\dot q_j-L$ is the Hamilton function $p_j\,=\,\partial L/\partial \dot q_j$. These functions satisfy the relations:

$$\frac{dg_j}{dt}\,=\,\frac{\partial H}{\partial p_j},\quad \frac{dp_j}{dt}\,=\,-\frac{\partial H}{\partial g_j} \tag{16}$$

Relations (16), which present a set of the Hamilton equations, are the closure conditions for exterior and dual forms [7]. They are similar to relations (5).}

The Schrődinger equation in quantum mechanics is an analog to field equation, where the conjugated coordinates are replaced by operators. The Heisenberg equation corresponds to the closure condition of dual form of zero degree. Dirac's *bra-* and *cket-* vectors made up a closed exterior form of zero degree. It is evident that the relations with skew-symmetric differential forms of zero degree correspond to quantum mechanics. The properties of skew-symmetric differential forms of the second degree lie at the basis of the electromagnetic field equations. The Maxwell equations may be written as $d\theta^2=0$, $d^*\theta^2=0$, where $\theta^2=\frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ (here $F_{\mu\nu}$ is the strength tensor). The Einstein equation is a relation in differential forms. This equation relates the differential of dual form of first degree (Einstein's tensor) and a closed form of second degree –the energy-momentum tensor. (It can be noted that, even Einstein's equation connects the closed forms of second degree, this equation is obtained from differential forms of third degree).

The connection the field theory equations with skew-symmetric forms of appropriate degrees shows that there exists a commonness between field theories describing physical fields of different types. This can serve as an approach to constructing the unified field theory. This connection shows that it is possible to introduce a classification of physical fields according to the degree of skew-symmetric differential forms. From relations (10) and (13) one can see that relevant degree of skew-symmetric differential forms, which can serve as a parameter of unified field theory, is connected with the degree p of evolutionary form in relation (10). It should be noted that the degree p is connected with the number of interacting balance conservation laws. {The degree of closed forms also reflects a type of interaction [6]. Zero degree is assigned to a strong interaction, the first one does to a weak interaction, the second one does to electromagnetic interactions, and the third degree is assigned to gravitational field.}

The connection of field-theory equations, which describe physical fields, with the equations for material media discloses the foundations of the general field theory. As an equation of general field theory it can serve the evolutionary relation (10), which is obtained the balance conservation laws for material media and has a double meaning. On the one hand, that, being a relation, specifies the type of solutions to equations of balance conservation laws and describes the state of material system (since it includes the state differential), and, from other hand, that can play a role of equations for description of physical fields (for finding the closed inexact forms, which describe the physical structures from which physical fields are made up). It is just a double meaning that discloses the connection of physical fields with material media (which is based on the conservation laws) and allows to understand on what the general field theory has to be based.

In conclusion it should be emphasized that the study of equations of mathematical physics appears to be possible due to unique properties of skew-symmetric differential forms. In this case, beside the exterior skew-symmetric differential forms, which are defined on differentiable manifolds, the skew-symmetric differential forms, which, unlike to the exterior forms, are defined on deforming (nondifferentiable) manifolds [7], were used.

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